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DRAG REDUCTION AND ADAPTIVE BOUNDARY CONDITIONS.(U)

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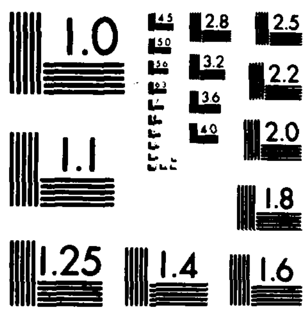

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This report considers the problem of increasing the maximum speed at which the boundary layer of a body in water can remain in laminar flow. It is shown that substantial improvement can be made by modifying the boundary conditions of the linear stability problem. The concern of affecting the turbulence itself to reduce drag is not addressed.

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PLAN OF THE REPORT

There are four sections of this report.

1.0 This gives a general overview of what is known and what is commonly done for the general viscous problem.

2.0 To obtain a qualitative feel for what may be achieved we turn in section 2.0 to the inviscid problem. The general statements that can be made about stability when boundaries are modified are described.

3.0 This section is devoted to specific inviscid examples. It is shown that dramatic changes in stability are achievable.

4.0 In section 4.0 an interesting stability problem which arises in boundary layer theory is analyzed.

1.0 GENERAL DISCUSSION

1.1 Introduction

The drag force on a body moving through the water is largely determined by the nature of a thin boundary layer between the stagnant fluid at the body surface and the free-stream fluid. At low speeds the flow in the boundary layer is smooth (laminar) and the drag is low. Beyond a well-defined critical speed, the flow becomes chaotic (turbulent) and the drag is much larger.

To reduce drag, one must manipulate the boundary layer in such a way that either the critical speed for transition to turbulence is increased beyond the desired operating speed or, given that the flow is turbulent, the turbulent flow is modified in a way which reduces the actual drag force. The classic motivation to study this problem is of course to increase the maximum speed of a vehicle with a given power plant. A second motivation, relevant to SSBN security, is the desire to reduce the intensity of the water disturbance caused by the passage of even a slowly moving vehicle.

Since turbulence is an extremely complicated and ill-understood phenomenon, we will not consider the problem of manipulating the turbulent boundary layer, despite its overwhelming practical importance. Our remarks are entirely directed to the problem of

increasing the maximum speed at which the boundary layer can remain in laminar flow, a problem which can be studied using the well-explored mathematics of linear stability theory. We will show that substantial improvements may be obtained by modifying the boundary conditions of the linear stability problem. This is by no means a new suggestion, but it seems to us that it has not been explored in sufficient generality, especially in view of the flexibility which modern electronic and sensor technology makes available.

The plan of this part of the report is as follows: in section 1.2 we summarize the relevant aspects of linear stability theory. In section 1.3 we discuss methods for manipulating the boundary layer profile about which the linear stability analysis is done. In section 1.4 we introduce the general notion of adaptive boundary conditions and their influence on linear stability.

1.2 The Linear Stability Problem

For the purpose of studying the drag due to skin friction we may limit our attention to essentially parallel flow past a flat plate. The flow field is then two-dimensional (we choose coordinates x parallel to the plate in the flow direction and y normal to the plate) and the relevant variables are the x and y components of velocity (u and v) pressure (P) and, if needed, temperature (T). The boundary conditions are usually that $u = v = 0$ at $y = 0$ and that $u = U_\infty$ at ∞. The boundary layer profile is governed by a diffusion-type equation in which x plays the role of time. This has the

consequence that the boundary layer thickness which we denote by δ, grows (typically as $\sqrt{x}$) as x increases. The actual profile depends on the precise conditions of the problem.

The linear stability problem is posed as follows: take a basic profile, $U_o(x,y)$, which varies so slowly with x that its x-dependence may be ignored. Then construct the equation of motion for small perturbations, $u_1(x,y)$, on that profile and look for solutions which are periodic in space and time (it is convenient to use the stream function of the perturbed flow

$$u_1 = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v_1 = -\frac{\partial\psi}{\partial x} \tag{1.1}$$

and look for solutions of the form $\psi(x,y) = \phi(y)e^{i\alpha(x-ct)}$.) The equation which emerges is the Orr-Sommerfeld equation,

$$\left(U(y)-c\right)\left(\phi''-\alpha^2\phi\right) - U''(y)\phi = \frac{i}{\alpha R}\left(\phi''''-2\alpha^2\phi'' + \alpha^4\phi\right) \tag{1.2}$$

where all velocities are measured in units of U_∞ [in particular $U(y) = U_o(y)/U_\infty$], and all lengths are measured in units of δ and $R = U_\infty\delta/\nu$ where ν is the viscosity of water. R is what is usually called the Reynolds number of the unperturbed flow. In order for ϕ to give an acceptable velocity perturbation, it must satisfy boundary conditions $\phi = \phi' = 0$ at both $y = 0$ and $y = \infty$. We are therefore faced with an eigenvalue problem: for given R and α we must solve

for c and its associated eigenfunction. The phase velocity c will in general be complex and the background flow is stable to small perturbations only if $\mathrm{Im}\, c < 0$ for all α (in that case, any small disturbance will decay away exponentially). A typical plot of stability regions is given in Fig. 1.1. The key generic feature is that for $R < R_{crit}$, disturbances of any wavelength decay with time and therefore the background flow is stable. For $R > R_{crit}$, some disturbances grow exponentially with time and the mean flow is probably unstable to transition to turbulence. Since δ grows as we move downstream the criterion for laminar flow over a plate of length L is that $U_\infty \delta(L)/\nu < R_{crit}$, where $\delta(L)$ is the boundary layer thickness at the downstream end of the plate. To enlarge the regime of laminar flow, it is obviously necessary to make R_{crit} as large as possible.

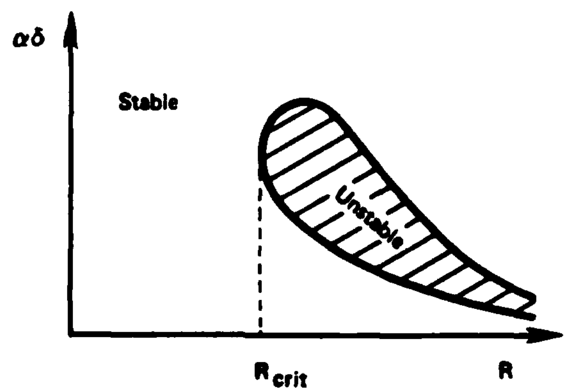

Figure 1.1

There are two methods of influencing R_{crit}, both of which we shall discuss. On the one hand, the value of R_{crit} depends on the details of the background profile, $U_0(y)$, and it may be possible to change the conditions of the problem so as to change $U_0(y)$ in a way which increases R_{crit}. On the other hand, one might take whatever background profile comes naturally but change the properties of the wall, thus changing the boundary conditions of the linear stability eigenvalue problem. We shall see that it is possible in this way to greatly change the stability of a flow. In what follows we will try to summarize what we know about both approaches.

1.3 Modifying the Boundary Layer

The equations which determine the downstream evolution of the boundary layer (the Prandtl equations) are obtained from the full Navier-Stokes equations by imposing the simplifying conditions that the flow is nearly parallel $(v \ll u)$ and that transverse variations are much more rapid than downstream variations $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$. If we include the possible effect of temperature variations on the viscosity, we obtain the system

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\nu(\tau) \frac{\partial u}{\partial y} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \bar{\nu} \frac{\partial^2 T}{\partial y^2} \qquad (1.3)$$

where ν is the kinematic viscosity of water and $\bar{\nu}$ is the thermal conductivity per unit mass.

The above equations are parabolic: given a profile at an initial value of x, the equations may be integrated forward in x to produce a predicted profile at any downstream x. Consequently, to specify the boundary layer one must specify an initial profile--i.e., one must know the details of how it started. This annoyance may be circumvented by looking for self-similar flows--flows in which the profiles at different downstream stations differ only in the value of a transverse dimensional scale. The Reynolds scaling property of these equations guarantee that we can find solutions in which the stream function, ψ, and temperature, T, have the form

$$\psi(x,y) = \sqrt{\nu x U_\infty} f(\eta)$$

$$T(x,y) = t(\eta)$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \quad . \tag{1.4}$$

The partial differential equations for the boundary layer then collapse to non-linear ordinary differential equations whose solution is completely determined by the boundary conditions at $y = 0$ and $y = \infty$. We will shortly discuss the properties of these solutions. There remains the question <u>why</u> we are interested in self-similar solutions: How do we know that the initial profile is just right to set up the

self-similar boundary layer? We will eventually show that the self-similar solutions have a certain stability property which guarantees that small deviations from the self-similar profile die away as we go downstream. We take this as a justification for concentrating on self-similar profiles for our discussion of the linear stability problem outlined in the previous section.

Let us first summarize the properties of the standard flat plate self-similar boundary layer in which temperature variations are ignored and the standard wall boundary conditions ($u = v = 0$ at $y = 0$) are applied. The equations to be solved are

$$2f''' + ff'' = 0 \tag{1.5}$$

$$u = U_\infty f'(\eta) \ , \ v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\,(\eta f' - f) \tag{1.6}$$

$$f = f' = 0 \quad \text{at} \quad \eta = 0 \tag{1.7}$$

$$f' = 1 \quad \text{at} \quad \eta = \infty \tag{1.8}$$

The equation for f must be solved numerically.

We need a definition of boundary layer thickness to specify a Reynolds number. There are two physically significant definitions:

$$\text{Displacement Thickness} = \delta_1 = \int_0^\infty dy\left(1-\frac{U}{U_\infty}\right) \tag{1.9}$$

$$\text{Momentum Thickness} = \delta_2 = \int_0^\infty dy\,\frac{U}{U_\infty}\left(1-\frac{U}{U_\infty}\right) \tag{1.10}$$

δ_1 is the net outward displacement of a far field streamline due to the presence of the plate. δ_2 measures the momentum lost from the flow due to the presence of the plate. For the self-similar flow at hand

$$\delta_1 = \sqrt{\frac{\nu x}{U_\infty}}\,[\eta - f(\eta)]_{\eta=\infty} = 1.72\sqrt{\frac{\nu x}{U_\infty}} \tag{1.11}$$

$$\delta_2 = \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty d\eta f'(1-f') = .664\sqrt{\frac{\nu x}{U_\infty}} \tag{1.12}$$

Both thicknesses grow as $\sqrt{x}$ as we move downstream from the front edge of the plate. It is customary to use δ_1 to construct the Reynolds number for the flow:

$$R_{\delta_1} = \frac{U_\infty \delta_1}{\nu} = 1.72\sqrt{\frac{U_\infty x}{\nu}} \tag{1.13}$$

The ratio δ_1/δ_2, which in this case has the value 2.59, turns out to be a convenient diagnostic quantity for stability of the flow.

The classic linear stability theory, when applied to this flow, gives a critical Reynolds number

$$\left(R_{\delta_1}\right)_{crit} = 420 \tag{1.14}$$

This is rather small--if we want absolutely stable flow over a surface 1 m long, the maximum flow velocity is 6 cm/s! This is due to the fact that Blasius flow is just on the boundary of instability. If the velocity profile has a point of inflection $\partial^2 U/\partial y^2 = 0$, the flow is potentially unstable; Blasius flow has a point of inflection precisely at $y = 0$. To increase stability it is obviously necessary to decrease the curvature at $y = 0$ by some means.

There are two effective methods for doing this: suction and heating. By suction is meant simply a change in the boundary conditions at $y = 0$ to $u = 0$, $v = -v_o$. This could be achieved in practice with porous walls and a pumping mechanism. With this boundary condition it is possible to find a solution in which the boundary layer is <u>independent</u> of x :

$$U = U_\infty \left(1 - e^{-v_o y/\nu}\right), \quad v = -v_o \tag{1.15}$$

Both δ_1 and δ_2 are now constants

$$\delta_1 = \frac{\nu}{v_o} \qquad \delta_2 = \frac{1}{2}\frac{\nu}{v_o} \tag{1.16}$$

and $R_{\delta_1} = U_\infty / v_o$ anywhere along the plate. The linear stability analysis shows that $R_{crit} = 50{,}000$, a factor 10^2 improvement over the no suction value. This is in part due to the fact that the curvature of the velocity profile is now everywhere negative and in part due to the fact that the diagnostic quantity δ_1/δ_2 has been reduced to 2 from

2.6. The practical possibilities are impressive: if one chooses $U_\infty/v_o = 10^4$, well within the stability region, it is possible to stabilize the boundary layer of a 5 m diameter 100 m long submarine moving at 20 kts by ingesting water at the trivial net rate of 3 m^3/s! The problem is that in order for this analysis to apply the suction must be applied uniformly over transverse dimensions smaller than the boundary layer thickness itself. Since the boundary layer is only a fraction of a millimeter thick this means that the boundary layer has to be ingested through small pores which will almost certainly clog in real ocean use. An alternate approach which may overcome this problem will be discussed at the end of this section.

The second stabilization method is to increase the temperature of the wall above that of the mean flow. Since the viscosity of water at room temperature decreases rather rapidly with increasing temperature, it becomes important to include the temperature in the boundary layer equations. If we evaluate the first boundary layer equation at the wall (where $u = v = 0$), we find:

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = -\left(\frac{1}{\nu}\frac{\partial \nu}{\partial y}\right)\left.\frac{\partial u}{\partial y}\right|_0 \qquad (1.17)$$

If the wall is hot, viscosity must increase outward as must u. Therefore $(\partial^2 U/\partial y^2)_0$ is less than zero, a condition we have already argued must increase stability.

Numerical integration of these equations for a heated wall have been carried out by several authors. It is found that $(R_{\delta_1})_{crit}$ increases dramatically with increasing heating, with a maximum value of 12,000 being obtained for 50°C temperature difference. This great increase in stability is not achieved at the cost of extreme modifications of the profile. What seems to count is the ratio δ_1/δ_2 : R_{crit} correlates very well with this quantity, increasing by 10^2 as δ_1/δ_2 decreases from 2.6 to 2!

Finally, we want to return to the problem of using suction to control the boundary layer. An alternative to distributed suction is to extract the boundary layer fluid through large, widely-spaced slots. The idea is that the boundary layer is allowed to grow for a distance D and then is sucked into an internal reservoir through a slot large enough to avoid clogging. A fresh boundary layer forms at the downstream edge of the slot and, after a distance D , is sucked in again. The maximum thickness of the boundary layer is evidently

$$\delta_{max} \sim 1.7\sqrt{\frac{\nu D}{U}} \tag{1.18}$$

In order for that thickness of fluid to be sucked in as it passes a slot of width d at stream velocity U , we must apply a suction velocity

$$\frac{v}{U} \sim 1.7\sqrt{\frac{\nu}{UD}}\,\frac{d}{D} \sim \frac{1}{R_{\delta_{max}}} \cdot \frac{d}{D} \tag{1.19}$$

The condition for stable flow along the plates of length D is that $R_{\delta_{max}} < 400$. This is achieved for interesting values of U_∞ for $D \sim 1$ m. We could realistically choose $d \sim 1$ cm and therefore control the boundary layer with $\frac{v}{U} < 10^{-4}$, which corresponds to a very manageable rate of water ingestion. The difficulty might be instability associated with the new type of shear layer across the slot itself. It is not clear that this shear layer is any more unstable than that along the wall, but we have seen no discussion of this problem. To repeat, the essence of this method is to achieve stability by interrupting the growth of the boundary layer every D meters so that the effective scale size for the stability problem is D rather than the overall vehicle length.

1.4 Adaptive Boundary Conditions

The discussion so far has dealt with situations in which the boundary conditions at the wall are fixed, and we have seen that large modifications in stability can be obtained with "small" changes in those boundary conditions. It might be more efficient to modify the wall conditions only in response to disturbances on the basic flow, trying to tailor the response in such a way as to damp the disturbance. If we are very lucky a scheme of this kind might allow us to obtain higher values of R_{crit} .

The basic situation is as described in section 1.2: We have a background flow described by a horizontal velocity profile $U_o(y)$. Small perturbations on this flow are described by a stream function

$\psi(x,y) = \phi(y)e^{i\alpha(x-ct)}$ which satisfies the Orr-Sommerfeld equation. We may eventually want to expand this system to include the effect of temperature variations or to allow the background profile to have inflow due to suction; but we will stick with the simplest system for explanatory purposes.

We propose to modify the nature of the eigenvalue problem for c by imposing a non-standard boundary condition at the wall (we of course retain the usual zero velocity condition, $\phi = \phi' = 0$, at $y = \infty$) in which the flow at the wall is a linear function of the past history of flow quantities sensed at the wall. Let $S(x,t)$ be the quantity sensed at the wall and let $\gamma(x,t)$ be the response. Then the sort of boundary condition we are contemplating is

$$\gamma(x,t) = \int_{-\infty}^{t} dt' \int dy G(x-y,t-t')S(y,t') \, . \tag{1.20}$$

Since the small disturbances are taken to depend on x and t as $e^{i\alpha(x-ct)}$, the above equation may be Fourier transformed:

$$\tilde{\gamma}(\alpha,\alpha c) = \tilde{G}(\alpha,\alpha c)\tilde{S}(\alpha,\alpha c) \, . \tag{1.21}$$

The only essential property of $\tilde{G}$ is that the singularities in αc lie in the lower half plane so that G is causal. G may reflect the mechanical properties of a material surface, in which case $\tilde{G}$ is prescribed, or it may be constructed via electronic coupling of sources and drivers, in which case it may be whatever we like. The general

problem will be to use the freedom to pick S and G to optimize stability. Subsets of this problem have been analyzed in some detail, but an overview seems to be lacking.

Let us summarize some of our options for sensing and responding. In terms of the perturbed stream function, $\psi = \phi(y)e^{i\alpha(x-ct)}$, the field quantities are

$$u = \phi'(y)e^{i\alpha(x-ct)} \tag{1.22}$$

$$v = -i\alpha\phi(y)e^{i\alpha(x-ct)} \tag{1.23}$$

$$\frac{i\alpha p}{\rho} = \left[\nu\left(-\alpha^2 + \frac{d^2}{dy^2}\right)\phi' + i\alpha U'\phi - i\alpha(U-c)\phi'\right]e^{i\alpha(x-ct)} \tag{1.24}$$

The wall is subject to transverse forces (due to pressure) or longitudinal forces (due to Reynolds stress, $\tau = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$) and either could be the quantity we want to sense. The sensed quantity will therefore be some linear combination of ϕ and its derivatives evaluated at the wall. The response quantity could be a motion of the wall itself (compliant wall) in either the normal or the transverse direction or a suction velocity without physical wall motion (adaptive suction). Again, the response quantity will be a linear combination of ϕ and its derivatives evaluated at the wall. If the wall moves as part of its response, the wall is no longer at $y = 0$, a fact which must be accounted for in writing out the boundary condition on ϕ. Clearly,

temperature or heat flux could be added in an appropriate way to this list.

We now list some specific boundary conditions of the type we are discussing. In all cases the problem is to solve the O-S equation in the interval $0 \leq y \leq \infty$ with $\phi(\infty) = \phi'(\infty) = 0$ plus the two specified conditions at $y = 0$:

Normal Compliant Wall:

$$c\phi'(0) + U'(0)\phi(0) = 0 \; ; \; i\alpha\phi(0) = \tilde{G}(\alpha,\alpha c)\, \frac{p(0)}{\rho} \tag{1.25}$$

Transverse Compliant Wall:

$$\phi'(0) = \tilde{G}(\alpha,\alpha c)\phi''(0) \; ; \; \phi(0) = 0 \tag{1.26}$$

Adaptive Suction (sensing pressure)

$$\phi'(0) = 0 \; ; \; i\alpha\phi(0) = \tilde{G}(\alpha,\alpha c)\, \frac{p(0)}{\rho} \tag{1.27}$$

Adaptive Suction (sensing Reynolds stress):

$$\phi'(0) = 0 \; ; \; i\alpha\phi(0) = \tilde{G}(\alpha,\alpha c)\phi''(0) \tag{1.28}$$

In all these cases the first condition refers to x components of velocity and the second to y components. $p(0)$ is meant to be evaluated in terms of ϕ. As advertised, the new boundary conditions involve linear combination of ϕ and its derivatives, plus an adjustable response Green's function which we have denoted by $\tilde{G}$.

2.0 THE EFFECT OF BOUNDARY CONDITIONS ON STABILITY OF FLOWS

2.1 Introduction

Many suggestions have been made to reduce the drag on bodies moving through fluids by delaying the transition from laminar to turbulent flow. Generically these ideas can be divided into two classes. One is to modify the flow in the vicinity of the body. Examples are heating, sucking, and the dispersal of polymers. The other is to change the nature of the surface presented by the body to the fluid. This can be done either actively or passively. Here we ask what can be said, very generally about the passive case--compliant boundary conditions.

As one of the simplest possible models we consider the case of inviscid, incompressible parallel shear flow. We imagine a flow $U(z)$ established between plates at z_1 and z_2 (z_2 may be infinity). The stability of the flow is then examined as a function of the boundary properties.

2.2 The Classical Theorems

The case in which the boundaries are both rigid plates has been extensively studied.

By virtue of Squire's theorem we can restrict attention to purely two dimensional perturbations. Thus we consider a perturbed stream function of the form

$$\psi = \phi(z)e^{i(\alpha x - ct)} \tag{2.1}$$

The stability problem reduces to finding (for fixed real α) when there are discrete eigenvalues $c = c_r + ic_i$ with $c_i > 0$ of the Rayleigh stability equation

$$(\partial_z^2 - \alpha^2)\phi - \frac{U''}{U - c}\phi = 0 \tag{2.2}$$

with the boundary conditions

$$\phi(z_1) = \phi(z_2) = 0 \tag{2.3}$$

Fortunately there are some (not many) general results known about this problem. These include:

(1) Rayleigh's Inflection Point Theorem.

This is: A <u>necessary</u> condition for <u>instability</u> is that the basic velocity profile should have an inflection point, i.e., there is a

$$z_s\ , z_1 < z_s < z_2 \quad \text{such that} \quad U''(z_s) = 0 \tag{2.4}$$

(2) <u>Fjortofts' Theorem</u>

A necessary condition for instability is that

$$U''(U-U_s) < 0 \tag{2.5}$$

somewhere in the flow. [Here $U_s \equiv U(z_s)$].

(3) Howard's Semi-Circle Theorem

For unstable waves c must lie in the semi-circle with radius $\frac{1}{2}(U_{max} - U_{min})$, center at $\frac{1}{2}(U_{max} + U_{min})$, and $c_i > 0$, i.e.,

$$[c_r - \tfrac{1}{2}(U_{max} + U_{min})]^2 + c_i^2 < \frac{1}{2}\left(\frac{U_{max} - U_{min}}{2}\right)^2 , \quad c_i > 0 \tag{2.6}$$

It should be emphasized that the Rayleigh Inflection Point Theorem is a necessary but not sufficient condition for instability. It is, however, of great qualitative importance. Even when considering viscosity most attempts to delay (or elimina-e) instability are devoted to getting as far away as possible from inflection points.

There are, however, a class of flows (which include most boundary layer flows) for which there also exists a sufficient condition for instability. Thus if

$$K(z) = U''/(U-U_s) \tag{2.7}$$

is regular at $z = z_s$ and

$$K(z) > \pi^2/(z_2-z_1)^2 \tag{2.8}$$

everywhere, then we have instability.

2.3 More General Boundaries

In the case of compliant boundaries the boundary conditions of Eq. (2.3) will no longer hold. The stability Eq. (2.2) will still be valid and since it is a second order differential equation we can put conditions at z_1 and z_2. As a rather general case we take these conditions to be that a linear combination of ϕ and its normal derivative be zero.

Thus

$$\phi(z_i) = + Y_i \frac{\partial\phi}{\partial n}(z_i) \tag{2.9}$$

(Note: The classical case is $Y_i = 0$. A free boundary corresponds to $|Y_i| \to \infty$).

Now let us see what happens to the previous theorems.

(1) The Inflection Point Theorem

Multiplying Eq. (2.2) by ϕ^*, integrating from z_1, to z_2 and using the boundary conditions, Eq. (2.9) gives

$$\int_{z_1}^{z_2} \left\{ \left|\frac{\partial\phi}{\partial z}\right|^2 + \alpha^2 |\phi|^2 + \frac{U''|\phi|^2}{U - c} \right\} dz = Y_1 \left|\frac{\partial\phi}{\partial z}\right|_1^2 + Y_2 \left|\frac{\partial\phi}{\partial z}\right|_2^2$$

$$\equiv \frac{1}{Y_1} |\phi|_1^2 + \frac{1}{Y_2} |\phi|_2^2 \; . \tag{2.10}$$

Let

$$Y_i = S_i + iT_i \ .$$

Then the imaginary part of Eq.(2.10) becomes

$$c_i \int_{z_1}^{z_2} \frac{U''|\phi|^2 dz}{|U-c|^2} = T_1\left|\frac{\partial\phi}{\partial z}\right|_1^2 + T_2\left|\frac{\partial\phi}{\partial z}\right|_2^2 \tag{2.11}$$

We note that in <u>general</u> the requirement

$$c_i > 0 \rightarrow \int_{z_1}^{z_2} \frac{U''|\phi|^2 dz}{|U-c|^2} = 0$$

only holds if $T_1 = T_2 = 0$. That is the Inflection Point Theorem holds only if $Y_i \equiv S_i$ <u>is real.</u> In this case

$$c_i = 0 \rightarrow U''(z_s) = 0 \ .$$

Until it is indicated differently we will be assuming Y_i is indeed real so that the Inflection Point Theorem holds.

(2) <u>Fjortofts' Theorem</u>

The real part of Eq. (2.10) is

$$\int_{z_1}^{z_2} \left\{ \left|\frac{\partial\phi}{\partial z}\right|^2 + \alpha^2|\phi|^2 + \frac{U''(U-c_r)|\phi|^2}{(U-c)^2} \right\} dz = S_1 \left|\frac{\partial\phi}{\partial z}\right|_1^2 + S_2\left|\frac{\partial\phi}{\partial z}\right|_2^2$$

(2.12)

Adding to this

$$0 = (c_r - U_s) \int_{z_1}^{z_2} \frac{U''|\phi|^2}{|U-c|^2} dz \tag{2.13}$$

gives:

$$\int_{z_1}^{z_2} \left\{ \frac{U''(U-U_s)|\phi|^2}{|U-c|^2} \right\} dz$$

$$= -\int_{z_1}^{z_2} \left\{ \left|\frac{\partial\phi}{\partial z}\right|^2 + \alpha^2|\phi|^2 \right\} dz + S_1 \left|\frac{\partial\phi}{\partial z}\right|_1^2 + S_2 \left|\frac{\partial\phi}{\partial z}\right|_2^2 \tag{2.14}$$

Thus in general Fjortoft's Theorem holds only if S_1 and $S_2 \leq 0$. Then a necessary condition for instability is

$$U''(U-U_s) < 0 \text{ somewhere in the flow.}$$

(3) <u>Howard's Circle Theorem</u>

This theorem essentially completely disappears when our more general boundary conditions are applied. Even the weaker Rayleigh theorem that c_r must be in the range of $U(z)$ fails to hold. To see this we parallel the usual derivation of the theorem.

Let

$$\phi^+ = \frac{\phi}{U-c} . \tag{2.15}$$

Then Eq. (2.12) becomes

$$\frac{\partial}{\partial z}(U-c)^2 \frac{\partial}{\partial z} \phi^+ - \alpha^2 (U-c)^2 \phi^+ = 0 \quad . \tag{2.16}$$

Multiply this equation by ϕ^{*+} and integrating yields:

$$\int_{z_1}^{z_2} \left\{ \left| \frac{\partial \phi^+}{\partial z} \right|^2 + \alpha^2 |\phi^+|^2 \right\} (U-c)^2 dz = \left[(U-c)^2 \phi^{*+} \frac{\partial \phi^+}{\partial z} \right]_{z_1}^{z_2} \tag{2.17}$$

Now *if* $\phi(z_1) = \phi(z_2) = 0$ then $\phi^+(z_1) = \phi^+(z_2) = 0$. (We are assuming $c_i > 0$ and hence $U-c \neq 0$.) The imaginary part of Eq. (2.17) becomes

$$2\, c_i \int_{z_1}^{z_2} \left\{ \left| \frac{\partial \phi^+}{\partial z} \right|^2 + \alpha^2 |\phi^+|^2 \right\} (U-c_r) dz = 0 \ , \tag{2.18}$$

and hence c_r is in the range of U. Consider, however, the general boundary condition

$$\phi(z_i) = S_i \left| \frac{\partial \phi}{\partial n} \right|_{z_i} , \ i = 1,2 \tag{2.19}$$

Then (2.17) becomes

$$\int_{z_1}^{z_2} \left\{ \left| \frac{\partial \phi^+}{\partial z} \right|^2 + \alpha^2 |\phi^+|^2 \right\} (U-c)^2 dz = \left\{ (U-c)^2 \left| \frac{1}{S_2} - \frac{U''}{U-c} \right| |\phi^+|^2 \right\}_{z_2} + \left\{ (U-c)^2 \left| \frac{1}{S_1} + \frac{U'}{U-c} \right| |\phi^+|^2 \right\}_{z_1} \tag{2.20}$$

Assuming $c_i \neq 0$ we obtain from the imaginary part of this equation that

$$\int_{z_1}^{z_2} \left\{ \left| \frac{\partial \phi^+}{\partial z} \right|^2 + \alpha^2 |\phi^+|^2 \right\} (U - c_r) dz$$

$$= \left\{ (U-c_r) \frac{|\phi^+|^2}{S_2} \right\}_{z_2} + \left\{ \frac{(U-c_r)|\phi^+|^2}{S_1} \right\}_{z_1}$$

$$- \left\{ \frac{U'}{2} |\phi^+|^2 \right\}_{z_2} + \left\{ \frac{U'}{2} |\phi^+|^2 \right\}_{z_1} \tag{2.21}$$

No particularly interesting result seems to follow from this.

2.4 Positive Results

So far all we have seen is that the introduction of our general boundary conditions weakens the few general theorems known about parallel shear flows. Now we will show that paralleling the proof of the sufficient condition for instability gives us a method to estimate the stability dependence on the S_i .

The essence of the approach is the following: We construct a neutrally stable solution for a particular α_s^2. Then one shows that for α^2 slightly lower than α_s^2 there is an unstable solution. Let $K(z) = -U''/(U-U_s)$ be regular at z_s . Set $c = c_s = U_s$.

The Rayleigh Equation (2.2) becomes

$$\phi'' + \left[K(z) - \alpha^2\right]\phi = 0 \tag{2.22}$$

This is a Sturm-Liouville problem with the variational principle

$$-\alpha = \min \frac{\int_{z_1}^{z_2} [|\phi'|^2 - K(z)|\phi|^2] dz - \frac{|\phi(x_2)|^2}{S_2} - \frac{|\phi(z_1)|^2}{S_1}}{\int_{z_1}^{z_2} |\phi|^2 dz} \tag{2.23}$$

First we show how this can be used to obtain a sufficient condition for instability and then how it determines dependence on the S_i.

$$\text{Let } \eta = \min \frac{\int_{z_1}^{z_2} f^2 dz - \frac{f(z_2)^2}{S_2} - \frac{f(z_1)^2}{S_1}}{\int_{z_1}^{z_2} f^2 dz} \tag{2.24}$$

Varying f we see that the minimum is obtained if

$$f'' + \eta f = 0$$

and

$$f(z_1) = -S_1 f'(z_1)$$

$$f(z_2) = S_2 f'(z_2) \tag{2.25}$$

Let η_m be the smallest eigenvalue of this problem. Then

$$\frac{\displaystyle\int_{z_1}^{z_2} |\phi'|^2\, dz - \frac{|\phi(z_2)|^2}{S_2} - \frac{|\phi(z_1)|^2}{S_1}}{\displaystyle\int_{z_1}^{z_2} |\phi|^2 dz} > \eta_m \tag{2.26}$$

therefore,

$$-\alpha_s^2 < \eta_m - \frac{\displaystyle\int_{z_1}^{z_2} K(z)|\phi|^2 dz}{\displaystyle\int_{z_1}^{z_2} |\phi|^2 dz} \tag{2.27}$$

Thus, if $K(z) > \eta_m$ everywhere

$$-\alpha_s^2 < 0$$

and so there is a neutrally stable mode. (Below we will calculate η_m for the special case $S_2 = 0$. It will be shown that

$$\eta_m = \left[\frac{\gamma 3\pi}{2(z_2 - z_1)}\right]^2 \tag{2.28}$$

where

$$0 < \gamma(S_1) < 1\ . \tag{2.29}$$

It can be shown that if there is such a neutrally stable mode, then there is an unstable mode for α^2 just less than α_s^2. This can be proved either by using perturbation theory[1] or more conveniently using the variational principle as demonstrated in the Appendix.

We can now see the effects of varying the S_i on stability. Suppose we have for some values of S_1, S_2 a neutrally stable mode (and consequently an unstable mode nearby). Then if by changing the S_i we can force α_s^2 to become negative we will achieve stability. But from Eq. (2.23) we readily find the change of α_s^2 with the S_i. Indeed since $-\alpha_s^2$ is stationary we obtain the rate of change of α_s^2 merely by differentiation with respect to the explicit dependence on the S_i. Thus

$$\frac{\partial}{\partial S_1}\left(-\alpha_s\right)^2 = \frac{1}{S_1^2}\,\frac{\left|\phi(z_1)\right|^2}{\int_{z_1}^{z_2} |\phi(z)|^2 dz} \tag{2.30}$$

and

$$\frac{\partial}{\partial S_2}\left(-\alpha_s\right)^2 = \frac{1}{S_2^2}\,\frac{\left|\phi(z_2)\right|^2}{\int_{z_1}^{z_2} |\phi(z)|^2 dz} \tag{2.31}$$

We see these slopes are always positive. Thus any increase in S_1 or S_2 always pushes $-\alpha_s^2$ to zero. The most stable situation is then that with $S_1, S_2 \to \infty$. The boundary conditions are then $\frac{\partial \phi}{\partial z_i} = 0$, i.e., a free boundary.

There is a caveat here, however. In general, the dispersion relation $[\alpha^2(c)]$ has many branches. Since we have implicitly assumed a continuous behavior, what has really been proved is that corresponding to a given branch $-\alpha_s^2$ always increased with increasing S_1, S_2. It may happen that corresponding to some values of the S_i new branches of the dispersion relation may appear--and a new instability may arise. All this is exemplified by the following example.

2.5 A Classic Example[1]

Consider $U = \sin z$, $z_1 < 0 < z_2$.

Then there is an inflection point at $z_s = 0$, and we can take $c_s = 0$. The stability equation becomes

$$\phi'' + (1-\alpha_s^2)\phi = 0 \tag{2.32}$$

For simplicity we will choose $S_2 = 0$, i.e., $\phi(z_2) = 0$. (2.33)

(Note: The η of the previous section is just $1 - \alpha_s^2$.) Using the boundary condition Eq. (2.33) and

$$\phi(z_1) = -S\left|\frac{\partial\phi}{\partial z}\right|_{z_1} \tag{2.34}$$

we find

$$\alpha_s^2 = 1 - \left(\frac{\lambda}{z_2 - z_1}\right)^2 \tag{2.35}$$

where λ is a solution of the equation

$$\tan \lambda = \frac{S_1}{z_2 - z_1} \lambda \quad . \tag{2.36}$$

Thus λ is the intersection of the curve $y = \tan \lambda$ with a straight line through the origin with slope $S_1/(z_2 - z_1)$.

Let us start with $\frac{S_1}{z_2 - z_1} \sim - \infty$.

The intersections occur at $(n + 1/2)\pi$, $n = 0 , 1 , . . .$

From Eq. (2.35)

$$\alpha_s^2 \sim 1 - \left[\frac{(n+1/2)\pi}{z_1 - z_2}\right]^2 \tag{2.37}$$

and we have instability only if

$$(n+1/2)\pi < (z_2 - z_1) \tag{2.38}$$

Clearly the mode $n = 0$ is the most unstable and the requirement for stability is

$$z_2 - z_1 < \pi/2$$

As $S_1/(z_2-z_1)$ increases from $- \infty$ to 0 the points of intersection move smoothly from $(n + 1/2)\pi$ to $(n + 1)\pi$. The $n = 0$ mode is still the

most unstable. At $S_1 = 0$ we have stability if $z_2 - z_1 < \pi$. Thus the range of $z_2 - z_1$ for which we have stability has doubled.

As $S_1(z_2 - z_1)$ grows from 0 to $+\infty$ intersection points move from $(n + 1)\pi$ to $(n + 3/2)\pi$. Again the most unstable of these is $n = 0$ and we apparently have stability for $z_2 - z_1 < \frac{3\pi}{2}$ . . This is, however, incorrect. For $S_1/(z_2 - z_1) = 1$ a new mode appears. Thus with $\lambda = 0$ we have a solution

$$\phi = z_2 - z \quad . \tag{2.39}$$

This is quite unstable with

$$\alpha_s^2 = 1 \quad . \tag{2.40}$$

When $S_1/z_2 - z_1)$ increases from 1 to $+\infty$ the intersection for this new mode moves from 0 to $\pi/2$. Thus in this range we have stability only for $(z_2 - z_1) < \frac{\pi}{2}$ and for part of the region $z_2 - z_1 << \frac{\pi}{2}$.

Combining the results we see that the most stability is obtained when $S_1/(z_2 - z_1) = 1 - \varepsilon$. Then we have stability for

$$(z_2 - z_1) < \pi(1 + \delta) \tag{2.41}$$

where

$$0 < \delta < \frac{1}{2} \quad ,$$

and

$$\delta \approx 1/4 \quad . \tag{2.42}$$

2.6 Complex Boundary Conditions

In the above we have found that for real boundary conditions some general global statements can be made. Thus, the Inflection Theorem holds, Fjortoft's theorem holds for $S_i \leq 0$, , and for a given branch of the dispersion curve increasing the S_i increases stability.

With complex boundary conditions we have not found any global theorems. However, we can make a local statement. Thus in the Appendix it is shown that in the vicinity of a neutrally stable solution there are two complex conjugate solutions for $\alpha < \alpha_s$. One of these is unstable. By an appropriate choice of complex Y_i it is possible to delay the instability.

We have the expressions

$$\left.\frac{\partial c}{\partial \alpha}\right|_{\alpha_s} = \rho_1 \pm i\ \rho_2 \tag{2.43}$$

and

$$-\alpha^2 = \frac{\int_{z_1}^{z_2} \left\{ |\phi'|^2 + \frac{U''}{U-c}\ |\phi|^2 \right\} dz - \frac{|\phi(z_2)|^2}{Y_2} - \frac{|\phi(z_1)|^2}{Y_1}}{\int_{z_1}^{z_2} |\phi(z)|^2 dz} \tag{2.44}$$

The change in $-\alpha^2$ at $-\alpha_s^2$ when $Y_i = S_i$ is replaced by $S_i + i\delta T_i$ is (since α_s^2 is stationary) obtained merely from the explicit change in the Y_i i.e.,

$$\delta(-\alpha_s^2) = \frac{1}{S_2^2} i\delta T_2 |\phi(z_2)|^2$$

$$+ \frac{1}{S_1^2} i\delta T_1 |\phi(z_1)|^2 \qquad (2.45)$$

i.e., $\delta\alpha_s = iB$ with B real.

Then

$$\delta c = (\rho_1 \pm i\rho_2)iB = iB\rho_1 \mp B\rho_2 \; . \qquad (2.46)$$

With the appropriate choice of sign of B we find

$$\operatorname{Im} \delta c < 0. \qquad (2.47)$$

Therefore, locally a complex Y_i (phase lag) can be chosen to stabilize.

2.7 Conclusion

It has been found that for real boundary conditions the stability for a given branch of the dispersion relation is always increased by increasing S_i (i.e., the closer to a free boundary one is the greater the stability). There is a warning, however. New, more unstable branches can arise.

Using complex boundary conditions can at least locally delay instability.

Hopefully in a following note we will endeavor to see what can be said when viscosity is included.

3.0 COMPLEX BOUNDARY CONDITIONS

3.1 Introduction

We have seen that when the classical rigid body boundary conditions are replaced by more general ones very little can be said in generality. Accordingly it is useful to consider some simple models. This is done here. It is found that dramatic changes in stability can be achieved.

The situation we envisage is that, instead of vanishing at the boundary ($z = z_i$), the stream function should satisfy

$$\phi(x,z_i) = \int_{-\infty}^{\infty} G(x-x') \frac{\partial\phi}{\partial n} (x',z_i)dx' \quad . \tag{3.1}$$

When Fourier transformed in the x coordinate this becomes

$$\phi(z_i) = Y_i(\alpha) \frac{\partial\phi}{\partial n} (z_i) \quad . \tag{3.2}$$

The reality condition on G implies

$$Y_i(-\alpha) = Y_i^*(\alpha) \quad . \tag{3.3}$$

Alternatively this can be written:

Let $$Y_i(\alpha) = S_i(\alpha) + iT_i(\alpha) \quad . \tag{3.4}$$

The reality condition is then that

$$S_i(-\alpha) = S_i(\alpha) \tag{3.5}$$

and

$$T_i(-\alpha) = - T_i(\alpha). \tag{3.6}$$

Since the Rayleigh stability equation is rather intractable for a general parallel shear flow we restrict ourselves to models in which the basic flow is at most piece-wise linear in z . However, discontinuities in U and its derivatives are admitted.

The problem then is to find those c such that

$$\frac{\partial^2\phi}{\partial z^2} - \alpha^2\phi = 0 \tag{3.7}$$

subject to the conditions

$$\Delta[(U-c)\phi' - U'\phi] = 0 \ , \tag{3.8}$$

$$\Delta\left[\frac{\phi}{U-c}\right] = 0 \ , \tag{3.9}$$

at discontinuities and

$$\phi(z_i) = Y_i \frac{\partial\phi}{\partial n}(z_i) , \qquad i = 1, 2 \tag{3.10}$$

at boundaries.

3.2 Model I

We consider the flow

$$U = 0 , \qquad 0 < z \leq d \tag{3.11}$$

$$U = Sz \qquad z > d \tag{3.12}$$

and the boundary conditions

$$\phi(0) = -Y(\alpha) \frac{\partial\phi}{\partial z}(0), \qquad \phi(\infty) = 0 . \tag{3.13}$$

The eigenvalue condition then is

$$c|\alpha| = \frac{S}{2} \left\{ 1 - \varepsilon^2 \frac{(1 + |\alpha|Y)}{(1 - |\alpha|Y)} \right\} \tag{3.14}$$

where $\varepsilon = \exp(-\alpha d)$.

In particular for the imaginary part of c we obtain

$$c_i = - S\varepsilon^2 \frac{\operatorname{Im} Y}{\left[1-|\alpha|Y\right]^2} \tag{3.15}$$

To incorporate the reality properties we write

$$Y = \frac{S'(\alpha)}{|\alpha|} + i\,\frac{T'(\alpha)}{\alpha}\ . \tag{3.16}$$

Then S' and T' must be even functions of α. The real growth rate αc_i is then given as

$$\alpha c_i = \frac{-\,S\varepsilon^2 T'}{\left[(1-S')^2 + T'^2\right]} \tag{3.17}$$

If we maximize with respect to T' we find

$$T' = 1 - S' \tag{3.18}$$

and then

$$\alpha c_i = \frac{-S\varepsilon^2}{1 - S'}\ . \tag{3.19}$$

Letting S' go to 1 from below or above (depending on the sign of S) we see that for all α we can obtain arbitrarily large damping. The corresponding Y is

$$Y = \frac{1}{|\alpha|} + \frac{i\ \mathrm{sgn}S}{\alpha}\,\eta \tag{3.20}$$

with η arbitrarily small and positive.

3.3 Model 2

This is similar to the above but slightly more complicated. We take boundaries at z_1 and z_2 with the same Y,

$$\phi(z_i) = Y(\alpha)\,\frac{\partial\phi}{\partial n}(z_i)\ , \qquad i = 1,\ 2. \tag{3.21}$$

The unperturbed flow is

$$U = U_1, \qquad z_1 < z < d + z_1 \tag{3.22}$$

$$U = U_1 + S(z - z_1 - d), \qquad z_1 + d < z < z_2 - d \tag{3.23}$$

$$U = U_2 = U_1 + S(z_2 - d), \qquad z > z_2 - d\ . \tag{3.24}$$

Since the flow is so symmetric we can divide the solutions into even and odd ones with respect to the midpoint which we take to be $z = 0$. Thus we merely need to look for solutions for $o < z < z_2$ subject to $\phi(0) = 0$ or $\phi'(0) = 0$. Also without loss of generality we can take $U_2 = 0$. We choose units so $z_2 = 1$.

The eigenvalue equation then becomes

$$c = \frac{S}{2|\alpha|}\,\frac{\left\{\left[1 - \frac{\lambda}{\varepsilon^2}\right][1+|\alpha|Y]e^{-2|\alpha|} - \left[\varepsilon^2 - \lambda\right][1-|\alpha|Y]\right\}}{[1+|\alpha|Y]e^{-2|\alpha|} - \lambda(1-|\alpha|Y)} \tag{3.25}$$

Here ε is again $\exp - |\alpha|d$ and $\lambda = +1$ for odd solutions and -1 for even ones.

This is somewhat complicated to discuss. Let us, however, just look at the region $|\alpha| >> 1,\ d \sim 1$.

The equation then becomes

$$c \approx -\frac{S}{2\alpha}\left|1 + e^{-2\alpha(1-d)}\frac{[1+|\alpha|Y]}{[1-|\alpha|Y]}\right| \qquad (3.26)$$

This then is just as for our first model. Therefore again we can achieve arbitrarily large damping.

3.4 Model 3: The Helmholtz Instability

Consider the flow in the region $z_1 < z < z_2$ where $z_1 < 0$, $z_2 > 0$. For $z_1 < z < 0$, $U(z) = -U_0/2$ and for $z_2 > z > 0$, $U(z) = U_0/2$. As boundary conditions we take

$$\phi(z_1) = 0\ , \qquad \phi(z_2) = Y\,\frac{\partial\phi(z_2)}{\partial z}\ . \qquad (3.27)$$

If $x = [U_0/2-c]/[U_0/2+c]$ the eigenvalue equation becomes

$$x^2 = \frac{\tanh[|\alpha|z_2] - |\alpha|Y}{\{\tanh[|\alpha|z_1]\}\{1-|\alpha|Y\tanh|\alpha|z_2\}} \qquad (3.28)$$

<u>Note:</u> If x is found, the c is given by

$$c = \frac{U_0}{2}\,\frac{(1-x)}{(1+x)}\ . \qquad (3.29)$$

First consider $Y \equiv 0$ (the usual Helmholtz case.) Then

$$x^2 = \frac{\tanh[|\alpha|z_2]}{\tanh[|\alpha|z_1]}\ . \qquad (3.30)$$

Since $z_2 > 0$, and $z_1 < 0$ we see x^2 is negative for all α . The two roots for x are purely imaginary and so one of them corresponds to an unstable mode.

Suppose now Y is real and

$$|\alpha| Y \to 1/\tanh\left[|\alpha| z_2\right] . \tag{3.31}$$

Since $\tanh^2\left(|\alpha| z_2\right) < 1$ we see that the numerator in the expression is negative while the denominator $\to -0$.

Therefore,

$$x^2 \to +\infty, \quad x = \pm\infty \tag{3.32}$$

and $c \to \frac{-U_0}{2}$ (real).

Thus stability has been achieved for all α by an appropriate choice of Y .

3.5 <u>Complex Boundary Condition and Boundary Layer Stability</u>

We have seen that the stability of flows in which the velocity is piecewise constant can be dramatically affected by the use of complex (in Fourier space) boundary conditions. Since our main interest is the stability of boundary layer flows, in which the velocity varies continuously, we would like to have some indication of what can be done in such cases.

We consider the Rayleigh equation for a flow $U(z)$ for which $U(0) = 0$, $U(\infty) = U_0$ and $U'(z) > 0$. If we denote the stream function by $\phi(z)e^{i\alpha(x-ct)}$ and define

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \psi_1 = e^{\alpha z}\phi \qquad \psi_2 = \frac{1}{\alpha}\, e^{\alpha z}\phi' \tag{3.33}$$

the Rayleigh equation may be written

$$\frac{d}{dz}\,\psi = \left[\alpha\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{U''}{\alpha(U-c)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\psi \tag{3.34}$$

with the boundary condition

$$\psi(\infty) = \begin{pmatrix}1\\ -1\end{pmatrix}. \tag{3.35}$$

The boundary condition at the wall $(z = 0)$ will be specified later. Since $U(z)$ is monotone, and has a finite range, it makes sense to use it as an independent variable. Then

$$\frac{d}{dU}\,\psi = \left[\frac{\alpha}{U'}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{U''/U'}{\alpha(U-c)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\psi \tag{3.36}$$

Now U' and U'' are to be thought of as functions of U.

We of course cannot solve this equation in general. Let us therefore consider the long wavelength limit, $\alpha \to 0$. We will see that it is easy to arrange the wall boundary conditions so that $\alpha c \to$ constant in this limit. The Rayleigh equation now reads

$$\frac{d\psi}{dU} = \frac{-U''/U'}{\alpha c}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\psi \tag{3.37}$$

and its solution (incorporating the boundary condition at $z = \infty$) is

$$\psi(U) = \exp\left\{\int_U^{U_o} dv\, \frac{U''/U'}{\alpha c}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$= \begin{pmatrix}1\\ -1 + \exp\displaystyle\int_u^{U_o} dv\, \frac{U''}{\alpha c U'}\end{pmatrix} \tag{3.38}$$

The boundary condition at the wall is expressed in general as

$$\frac{\psi_2(0)}{\psi_1(0)} = \frac{\phi'(0)}{\alpha\phi(0)} = \frac{\tilde{G}(\alpha,\alpha c)}{\alpha} \tag{3.39}$$

where $\tilde{G}$ has to satisfy certain reality and causality conditions. A form which is general enough for our purpose is

$$\frac{\tilde{G}}{\alpha} = \frac{1}{\left(a\,\dfrac{\alpha c}{U'(0)}\right) - ib} \tag{3.40}$$

where a and b are dimensionless, $\frac{a}{b} > 0$ and we have introduced $U'(0)$ to nondimensionalize αc .

The eigenvalue condition for αc is now

$$-1 + \exp\int_0^{U_0} dv\, \frac{U''/U'}{\alpha c} = \frac{1}{\left(a\,\dfrac{\alpha c}{U'(0)}\right) - ib} \tag{3.41}$$

The integral in the exponent can be done and we obtain [recalling that for a boundary layer flow $U'(\infty) = 0$],

$$\exp\left(-\frac{U'(0)}{\alpha c}\right) = 1 + \frac{1}{\left(a \frac{\alpha c}{U'(0)}\right) - ib} \tag{3.42}$$

If, for simplicity, we set $a = 0$, we have

$$\frac{\alpha c}{U'(0)} = -\frac{1}{\ln(1 + ib)}, \qquad b \text{ arbitrary.} \tag{3.43}$$

The imaginary part of αc may be made as large and negative as we like by taking b small and negative.

What does this mean? A boundary layer with definite sign of U'' is inviscidly stable anyway. On the other hand, the decay rate is expected to go to zero as $\alpha \to 0$, indicating a nearby instability. With the new boundary condition we can push this instability away. A profile with an inflection point may be inviscidly unstable for small enough α . We apparently can control this instability as well. We really want to know whether we can improve stablity uniformly for all α , and the above discussion gives only a hint that this is possible.

3.6 Conclusion

From these simple models we conclude it should be possible by constructing appropriate boundary materials to dramatically affect stability.

4.0 BOUNDARY LAYER PERTURBATION THEORY

We would like to pose a problem which arises in the study of the flat plate boundary layer. The basic equations (the Prandtl equations) for the velocity components (u is parallel to the plate, v , perpendicular) are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

with the boundary conditions

$$u(0) = v(0) = 0 \ , \quad u(\infty) = U_0 \quad v(\infty) = 0$$

This system is parabolic: given an initial profile $u(x_0,y)$ one integrates forward in x to obtain $u(x,y)$ at any later x. The standard Blasius solution is obtained by imposing the condition of self-similarity

$$u(x,y) = F(y\sqrt{u_0/\nu x}) \ ,$$

a condition which can be met because of the Reynolds scaling property of the equations. Suppose the initial conditions deviate from those required to produce a self-similar solution. What happens then? Does the solution eventually converge to Blasius, or does it perhaps evolve away from it?

To attack this question we will try to pose the linear stability problem for small perturbations around the Blasius solution to the Prandtl equations. The equations are best stated in terms of the stream function ψ ,

$$u = \frac{\partial\psi}{\partial y} \quad v = -\frac{\partial\psi}{\partial x} \quad . \tag{4.1}$$

A self-similar profile must result from a stream function of the form

$$\psi = \sqrt{\nu U x}\, f_o(\eta) \qquad \eta = y\sqrt{\frac{U}{\nu x}} \, .$$

The Prandtl equations then reduce to an ordinary differential equation

$$f_o''' + 2f_o f_o'' = 0 \tag{4.2}$$

$$f_o(0) = f_o'(0) = 0, \; f_o'(\infty) = 1 \quad ,$$

whose solution can be obtained numerically.

We want to study the time evolution of small perturbations on f_o. To that end we consider stream functions of the general form

$$\psi = \sqrt{\nu U x}\left(f_o(\eta) + f_1(\eta)x^a\right) \, . \tag{4.3}$$

and expand the Prandtl equations to first order in f_1 . We find

$$f_1''' + \frac{f_o}{2} f_1'' - af_o' f_1' + \left(a + \frac{1}{2}\right) f_o'' f_1 = 0 \qquad (4.4)$$

This is to be solved subject to boundary conditions guaranteeing that f_1 does not change the basic flow:

$$f_1(0) = f_1'(0) = 0, \; f_1'(\infty) = 0 \qquad (4.5)$$

For the moment the parameter a, which governs the downstream growth rate of the perturbation, is arbitrary. In fact, the differential equation plus boundary conditions pose a sort of eigenvalue problem for a. If we find that $\mathrm{Re}(a) < 0$, then we will conclude that perturbations on the standard Blasius profile die away downstream, a physically satisfactory result.

To see whether the boundary conditions can be met, we look at the large y limit of Eq. (4.4). That, in turn, is governed by the large y limit of $f_o(\eta)$. The asymptotic behavior of f_o is found from a study of Eq. (4.2) to be

$$f_o(\eta) \rightarrow \eta - 1.72 + O\left(e^{-\eta^2/4}\right) . \qquad (4.6)$$

Substituting this into Eq. (4.4) and dropping terms of $O\left(e^{-\eta^2/4}\right)$ we obtain

$$f_1''' + \frac{1}{2}(\eta - 1.72)\, f_1'' - af_1' = 0 .$$

This is a standard hypergeometric equation for f_1 with two non-trivial solutions. The three possible asymptotic behaviors of f_1 are

$$f_1 \sim 1, \ \left(\eta^2\right)^{a+1/2}, \ e^{-\frac{\eta^2}{4}}\left(\eta^2\right)^{-(a+1)} \tag{4.7}$$

All solutions satisfy the boundary conditions of Eq. (4.5) so long as $a + 1/2 < 1$. (We will shortly see that we must actually impose the condition $a + 1/2 < 0$) . For a less than this limit, any a is acceptable. For a greater than this limit, only discrete values of a for which the coefficient of the $(\eta^2)^{a+1/2}$ solution vanishes will be acceptable.

To see why we must impose the condition $a + 1/2 < 0$, it is helpful to look at the two important definitions of boundary layer thickness-displacement thickness (δ_1) and momentum thickness (δ_2) . The definitions are

$$\delta_1 = \int_0^\infty dy\left(1 - \frac{u}{U_\infty}\right)$$

$$\delta_2 = \int_0^\infty dy\, \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) \tag{4.6}$$

where u is the component of velocity parallel to the wall. δ_1 is equal to the outward displacement of the stream lines far from the wall compared to their position when the wall is absent. δ_2 is the thickness which would have to be removed from the free stream flow in order to match the momentum actually lost to the wall. If we use our

general expression for ψ [Eq. (4.3)], remember that $u = \partial\psi/\partial x$, expand to first order in f_1 and integrate by parts whenever necessary, using the equations satisfied by f_o and f_1, we obtain

$$\delta_1 = \left[1.72 + x^a\left(-f_1(\infty)\right)\right]\sqrt{\frac{\nu x}{U_\infty}}$$

$$\delta_2 = \left[.66 + x^a \frac{f_1''(0)}{a+1/2}\right]\sqrt{\frac{\nu x}{U_\infty}} \qquad (4.7)$$

Since the displacement thickness must be finite on physical grounds, it is apparent that we must impose the condition $f_1(\infty) < \infty$. By the previous paragraph, this means that we must choose $a + 1/2 < 0$ or find a (hypothetical) discrete eigenvalue of a for which the offending $(\eta^2)^{a+1/2}$ term does not appear in the asymptotic behavior of f_1. Since a numerical search did not show any sign of the existence of a (real) eigenvalue, we shall henceforth ignore this possibility.

As has been noted in many places, the parameter which is of the greatest importance for stability is the ratio δ_1/δ_2. A 10% change in this parameter can cause an order of magnitude change in the critical Reynolds number. To first order in f_1 we have

$$\frac{\delta_1}{\delta_2} = 2.59\left[1 - x^a c\left(\frac{\Phi(a)}{1.72} + \frac{1}{.66(a+1/2)}\right)\right]$$

$$\Phi(a) = \frac{f_1(\infty)}{f_1''(0)}, \qquad c = f_1''(0)$$

$f_1''(0)$ can be thought of as the constant normalizing f_1: one integrates Eq. (4.4) with the starting conditions $f_1(0) = f_1(0) = 0$ $f_1''(0) = c$. The quantity $\Phi(a)$ is obtained by integrating the differential equation out to infinity, and is just a function of a , defined for $a < -1/2$.

We have done the requisite numerical integration for a range of values of $a < -1/2$ and we find that by choosing the sign of c properly we can always arrange the perturbation to decrease the quantity δ_1/δ_2 thereby increasing stability. Because $a < -1/2$, the perturbation must die away downstream at least as rapidly as $x^{-1/2}$. Since $x^{-1/2}$ is a rather slow rate of decrease, there might be some advantage to attempting to set up the perturbation for which $a = -1/2$ --the stability increase which is achieved then decays very slowly, perhaps slowly enough to maintain laminar flow over a region of interesting size.

The issue we have not yet explored is the question of whether there may be complex discrete eigenvalues and whether to represent a general perturbation, complex values of a are required. We would like to come back to this question in the future, although we expect it to be a rather difficult question to analyze, since the operator involved is not hermitean.

Figure 4.1

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APPENDIX

Here we want to show that there is an unstable mode for α slightly smaller than α_s. It was shown in Eq. (2.23) that

$$-\alpha^2 = \frac{\int_{z_1}^{z_2} \left\{ |\phi'|^2 + \frac{U''}{U-c} |\phi|^2 \right\} dz - \frac{|\phi(z_2)|^2}{S_2} - \frac{|\phi(z_1)|^2}{S_1}}{\int_{z_1}^{z_2} |\phi|^2 dz} \tag{A1}$$

Consider the change of $-\alpha^2$ with c in the vicinity of c_s. Since $-\alpha_s^2$ is stationary we merely need to differentiate with respect to the explicit occurrence of c. Since we are looking for unstable modes we evaluate at $c = c_s + i\varepsilon$

$$-\left.\frac{\partial\alpha^2}{\partial c}\right|_{\alpha_s} = \frac{\int_{z_1}^{z_2} \frac{U''|\phi|^2}{(U-c-i\varepsilon)^2} dz}{\int_{z_1}^{z_2} |\phi|^2 dz} \tag{A2}$$

Since $U''(z_S) = 0$ the integrand in the numerator of Eq. (A2) has a simple pole at $z = z_s$. The $i\varepsilon$ tells how we are to go around this. We find

$$-\left.\frac{\partial\alpha}{\partial c}\right|_{\alpha_s} = \frac{P\int_{z_1}^{z_2}\frac{U''|\phi|^2 dz}{U-U(z_s)} + \frac{\pi i\,[\operatorname{sgn} U'(z_s)]\,U'''(z_s)\left|\phi^+(z_s)\right|^2}{[U'(z_s)]^2}}{\int_{z_1}^{z_2}|\phi(z)|^2 dz} \tag{A3}$$

i.e.,

$$-2\alpha\left.\frac{\partial\alpha}{\partial c}\right|_{\alpha_s} = a + ib\ . \tag{A4}$$

therefore,

$$\left.\frac{\partial c}{\partial\alpha}\right|_{\alpha_s} = \frac{-2\alpha_s(a-ib)}{a^2+b^2} \tag{A5}$$

with

$$a = \frac{P\int_{z_1}^{z_2}\frac{U''|\phi|^2 dz}{U-U(z_s)}}{\int_{z_1}^{z_2}|\phi(z)|^2 dz} \tag{A6}$$

and

$$b = \frac{\pi U'''(z_s)|\phi(z_s)|^2 \operatorname{sgn} U'(z_s)}{[U'(z_s)]^2\int_{z_1}^{z_2}|\phi(z)|^2 dz} \tag{A7}$$

Then for small $\alpha - \alpha_s$

$$c_i = \frac{2\alpha_s b}{a^2 + b^2}\left(\alpha - \alpha_s\right) .$$

Let us see how this checks with our example.

$$U(z) = \sin z, \qquad z_s = 0 .$$

Then $U'(z_s) = +1$, $\quad U'''(z_s) = -1$.

Therefore, $\quad b < 0$.

If $\alpha > \alpha_s$ Eq. (A8) gives $c_i < 0$, contrary to our assumption. On the other hand if $\alpha < \alpha_s$ we see we do indeed obtain a $c_i > 0$. We conclude there is an unstable mode for α slightly smaller than α_s !

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